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Exploiting Motor Modules in Modular Contexts

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Summary. Recently there has been a growing interest in modeling motor control systems with modular structures. Such control structures have many interesting properties, which have been described in recent studies. This chapter focuses on some properties which are related to the fact that specific set of contexts can themselves be modular. Specifically, it shows that the adaptation of a modular control structure can be guided by the modularity of contexts, by means of interpreting the current unexperienced context as the combination of previously experienced contexts.

1.1 Introduction

Humans exhibit a broad repertoire of motor capabilities which can be performed in a wide range of different environments and situations. From the point of view of control theory, the problem of dealing with different environmental situations is nontrivial and requires significant adaptive capabilities. Even the simple movement of lifting up an object, depends on many variables, both *internal and external* to the body. Examples of internal variables are the state of the arm (e.g. joint angles and angular velocities) and its dynamic parameters (e.g. masses, moments of inertia, etc.). However, when interacting with the environment, motor commands need to be adapted also to some *external variables* which describe the interaction environment (e.g. geometrical and dynamical parameters of lifted objects). All these variables define what is generally called the context of the movement. As the context of the movement alters the input-output relationship of the controlled system, the motor command must be tailored so as to take into account the current context. In everyday life, humans interact with multiple different environments and their possible combinations. Therefore, a fundamental question in motor control concerns how the control system robustly adapts to a continuously changing operating context.

The adaptation of the motor control system to a continuously changing environment is a complex task. Noticeably, the dimensionality of the context space grows exponentially with the number of alternatives/variables used to describe the context itself. Therefore, this *curse of dimensionality* rules out any brute force approach for solving the context adaptation problem. Moreover, recent studies have shown that nature have found much more interesting solutions. Specifically, it has been proposed that the adaptation to new contexts can be achieved by means of linearly combining the knowledge about previously experienced contexts. In the following section this principle is described in details by discussing some related experimental evidence.

1.1.1 Experimental evidence on modular organization of the motor control system

Recently, a huge amount of literature have shown that biological motor control systems are based on a continuous adaptation of internal models. These models can be seen as an internal representations used by the central nervous system to handle a variety of different contexts. Within this framework, there has been recently a major interest in modeling these internal models by means of linear combinations of a finite number of elementary modules. According to this modular architecture, multiple controllers co-exist, with each controller suitable to a specific context. If no controller is available for a given context, the individual controllers can be combined to generate an appropriate motor command. Therefore, even if internal models could be learned by a single module, there seems to be at least three main advantages [Wolpert and Kawato, 1998] related to their organization in multiple modules:

- **Modularity of contexts.** The contexts within which the model operates can be themselves modular. Experiences of past contexts and objects can be meaningfully combined; new situations can be often understood in terms of combinations of previously experienced contexts.
- **Modularity of motor learning.** In a modular structure only a subset of the individual modules cooperate in a specific context. Consequently, only these modules have a part in motor learning, without affecting the motor behaviors already learned by other modules. This situation seems more effective than a global structure where a unique module is capable of handling all possible contexts. Within such a global framework, motor learning in a new context possibly affects motor behaviors in other (previously experienced) contexts.
- **Dimensionality of generated motor acts.** By modulating the contribution of each module, a huge amount of new behaviors can be generated. Modules can be seen as a vocabulary of motor primitives, which are the building blocks for constructing complex novel motor acts.

Remarkably, there is experimental evidence supporting the hypothesis that biological motor control systems are organized in modules. In particular, Mussa-Ivaldi and Bizzi [Mussa-Ivaldi and Bizzi, 2000] have shown that this modularity is already present at the spinal cord level in the form of multiple goal directed muscle synergies acting at the level of individual limbs. Each synergy has been described in terms of the force field generated at the extremity of the limb. Interestingly, the observed fields were limited in number (activated fields were grouped into few classes), goal directed (the force field converged toward an equilibrium point) and linearly combinable (the simultaneous stimulation led to vector summation of the generated forces). In our opinion this experiment, though limited to frogs and rats, paves the way to interpret biological motor control systems (and the associated internal models) as modular structures.

On a different context, experiments with monkeys have shown that the premotor cortex has a modular structure and in particular area F5 has been shown to contain neurons responding to the execution of different grasp types. They have been typically regarded as constituting a vocabulary of motor acts, with populations of neurons coding and controlling the execution of a particular grasp (e.g. precision grip, full palm grasp, etc.) [Fadiga et al., 2000].

Concerning experiments with humans, there has been a growing interest in understanding if internal models are organized modularly within the central nervous system. The existence and the adaptability of kinematic [Flanagan and Rao, 1995] and dynamic internal models [Shadmehr and Mussa-Ivaldi, 1994] have been extensively proved. These two representations have been proved to be weakly intertwined [Krakauer et al., 1999], thus revealing a first level of modularity.

A second level of modularity is revealed when considering human capability of switching between previously learned internal models. Specifically, although the time course of adaptation to a novel context can extend over hours, restoring the pre-perturbation behavior is often faster [Welch et al., 1993, Brashers-Krug et al., 1996] thus suggesting the restoration of the previously acquired module.

At a third level of understanding, there is also evidence suggesting that previously acquired modules can be generalized to new situations if the novel context can be interpreted as the combination of previously experienced contexts. In particular, in a kinematic scenario, it has been shown that different visuomotor mappings can be learned [Ghahramani and Wolpert, 1997] and these maps can be interpolated to create new ones. Similarly, in a dynamic scenario, human subjects have shown the ability of combining previously acquired internal models when new contexts can be interpreted as the combination of previously acquired contexts. Specifically, after learning the correct grasping forces for lifting two different objects, subjects have displayed the ability of producing the correct force for lifting both objects simultaneously without any training [Davidson and Wolpert, 2004].

As a concluding remark, the robustness displayed by biological motor control systems seems to be the result of an extraordinary capability of adapting to continuously changing contexts. This adaptation seems to be achieved in a twofold manner: (1) by a continuous update of existing modules (2) by the combination (switching) of (between) previously learned modules. Remarkably, experiments suggest that the two processes take advantage of different information: while adaptation can be attributed to performance errors [Shadmehr and Mussa-Ivaldi, 1994], switching and combination depend on sensory components of the context [Shelhamer et al., 1991].

1.1.2 Adaptive and modular motor control strategies

Based on these findings, there has been recently a growing interest in investigating the potentialities of *adaptive and modular* control schemes as proposed in [Wolpert and Kawato, 1998] and [Mussa-Ivaldi, 1997]). These investigations suggest the possibility of developing humanoid robots capable of adapting to a continuously changing environment. At the current state of the art, performance errors based adaptation have been extensively studied [Slotine and Li, 1991] and [Sastry and Bodson, 1989]. However, modern robots still miss the capability of exploiting previous experiences on the basis of context related sensory information. Considering our previous discussion, modularity seems to play a fundamental role within this framework. Remarkably, modularity does not seem to be useful *per se* but reveals its usefulness when adapting to modular contexts. Therefore, the remaining of the chapter aims at giving further evidence to the fact that a modular control structure is extremely useful when associated to a set of modular contexts.

Within these investigations, modularity is formalized in terms of multiple forward/inverse models[3]. Motor commands are usually obtained by combining these elementary internal models: different combinations serve different contexts. Assuming that a set of possible operational contexts has been defined, two fundamental questions must be faced:

1. Is there a way to choose the elementary internal models so as to cover all the contexts within the specified set?
2. Given a set of internal models which appropriately cover the set of contexts, how is the correct subset of internal models selected for the particular current context?

Both questions have been already investigated in [Wolpert and Kawato, 1998] and in [Mussa-Ivaldi and Giszter, 1992] within the function approximation framework. Recently, the same two questions have been considered [Nori and Frezza, 2005] within a control theoretical framework. So far this novel approach has been proved to provide interesting results in answering the first of the two questions as shown in [Nori and Frezza, 2004a] and [Nori, 2005]. This work proceeds along the same line to answer the second question.

Having in mind an application in the humanoid robotic field, the present work proposes a strategy to adaptively select a given set of inverse models. The selection process is based both on performance errors (Section 1.3.3) and context related sensory information (Section 1.3.4). Remarkably, both these quantities have been shown to play a fundamental role in the the module adaptation and selection processes (see Section 1.1.1). The key features of the proposed control scheme are the following:

- **Minimum number of modules.** Previous works [Nori, 2005] have established the minimum number of modules which are necessary to cover all the contexts in a specified set. The present paper will describe how this minimality result can be fitted in the adaptive selection of the modules.
- **Linear combination of modules.** The theory of adaptive control has been widely studied since the early seventies. Interesting results have been obtained, especially in those situations where some linearity properties can be proved and exploited. In our case, linearity will be a property of the considered set of admissible contexts.

The remaining sections are organized as follows. Section 1.2 gives our formal definition of modular motor control strategy; a simple example is analyzed and a solution is given. Section 1.3 describes a similar scenario but immersed in different contexts; a modular architecture is proposed as a possible solution to the problem of constructing the correct control strategy according to the current context (Section 1.3.1). Finally, two intertwined solutions to the context adaptation problem are proposed: a performance error based adaptation (Section 1.3.3) to be used when nothing is known about the current context and a “error free” adaptation which instead relies on a priori context related sensory information (Section 1.3.4).

[3] Here a forward model is considered to be a map from motor commands to the corresponding movement. Viceversa, an inverse model corresponds to a map from desired movement to motor commands.

1.2 Reaching with a modular control structure

This section gives a formal definition of the motor control modules. In a mathematical framework, the system to be controlled will be described by the following differential equation:

$$\dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\mathbf{u} \tag{1.1}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the system state and the vector $\mathbf{u} \in \mathbb{R}^m$ is the system input, corresponding to our control variable. Equation (1.1) describes the state evolution given the current input; its internal representation correspond to what is usually called forward internal model. An inverse internal model instead, defines the input to be used in order to obtain a desired state evolution.

Different definitions of modular control strategies can be given; the present chapter follows the formalization which was originally proposed as a mathematical description of the experimentally observed spinal fields [Mussa-Ivaldi and Bizzi, 2000]. Practically speaking the original control variable $\mathbf{u}$ is replaced by the linear superposition of a finite number of elementary control strategies $\{\Phi^1(\mathbf{x}), \Phi^2(\mathbf{x}), \dots, \Phi^K(\mathbf{x})\}$. Specifically:

$$\mathbf{u} = \sum_{k=1}^{K} \lambda_k \Phi^k(\mathbf{x}), \tag{1.2}$$

where the new control variables are the mixing coefficients $\lambda_1, \lambda_2, \dots, \lambda_K$ assumed to be constant during the execution of a single movement.

To exemplify the proposed ideas it results convenient to consider a specific task, nominally reaching, i.e. moving a limb to desired final point. Within this framework, the system to be controlled will be a mathematical model of the limb dynamics, which can always be written as follows:

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + g(\mathbf{q}) = \mathbf{u}, \tag{1.3}$$

where $\mathbf{q}$ are the generalized coordinates which describe the pose of the kinematic chain (e.g. joint angles), $\mathbf{u}$ are the control variables (e.g. torques applied at the joints) and the quantities M, C and g are the inertia, Coriolis and gravitational components. Remarkably, (1.3) can always be written in the state space form (1.2) defining $\mathbf{x} = [\mathbf{q}, \dot{\mathbf{q}}]$.

The mathematical formulation of reaching consists in finding a time varying input $\mathbf{u}$ that drives the system configuration $\mathbf{x}$ to any desired reachable state $\mathbf{x}_f$. If our control input were the original (time variant) input variable $\mathbf{u}$, then the problem would be easily solved by using classical control techniques [Murray et al., 1994]. In the modular structure framework, the input variables are the (time invariant) mixing coefficients $\lambda_1, \lambda_2, \dots, \lambda_K$ and therefore a major attention should be posed in selecting the modules. If these functions Φ_k are not chosen properly, some configurations (reachable by a suitable choice of the original input $\mathbf{u}$) might be no longer reachable by the new input variables. Formally speaking, the system (1.3) might loose its controllability by imposing the modular structure (1.2) on its input. As to this concern, the following problem has been formulated:

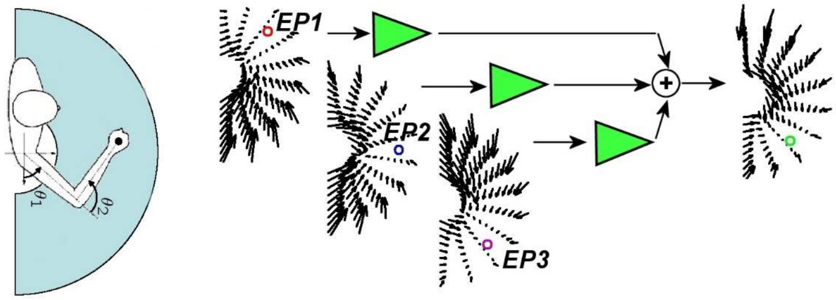

Fig. 1.1. A graphical representation of the modular reaching problem. A finite number of reaching movements (each with a specific equilibrium point, indicated with EP) are used as primitives; these movements are represented with the associated force fields. Given an arbitrary point to be reached (indicated in green) the primitives needs to be linearly combined so as to obtain a new force field capable of reaching the desired point. Notice that the linear mixing coefficients (the triangles) are represented with a green color so as to make explicit their dependence on the desired reaching point.

Problem 1 (Synthesis of Elementary Controls for Reaching). Find a set of modules $\{\Phi^1, \dots, \Phi^K\}$ and a continuously differentiable function $\lambda(\cdot)$, such that for every desired reachable state $\mathbf{x}_f$ the input:

$$\mathbf{u} = \sum_{k=1}^{K} \lambda_k(\mathbf{x}_f)\Phi^k(\mathbf{x}) \tag{1.4}$$

steers the system (1.3) to the state $\mathbf{x}_f$ (see also Figure 1.1).

As previously pointed out, previous attempts to solve the above controllability problem were based on an approximation framework [Wolpert and Kawato, 1998] and [Mussa-Ivaldi and Giszter, 1992]. The main drawback of this approach is the high number of required modules. Such a drawback follows from a quite general principle: without any a priori knowledge on the function to be approximated, the more the modules are, the better the approximation will be. This work follows a different solution originally proposed in [Nori and Frezza, 2004b] and based on the idea of maintaining the number of primitives as low as possible. Remarkably, the proposed solution is composed by $n + 1$ primitives[4] which is proved to be the minimum number under suitable hypotheses (see Section A). Details on how to construct such a minimal solution to Prb. 1 can be found in [Nori and Frezza, 2005].

1.3 Reaching in different contexts

In order to immerse the reaching action into different contexts, let's now consider reaching while holding objects with different masses and inertias. The underlying

[4] Remember that n is the state space dimensionality, i.e. $\mathbf{x} \in \mathbb{R}^n$.

idea is to replicate a framework similar to the one proposed in different human behavioral studies [Davidson and Wolpert, 2004, Krakauer et al., 1999]. Within such framework, a successful execution of the reaching movements requires a control action which should compensate for the perturbing forces. Since the controlled system[5] changes its properties with the context, suitable changes should be imposed on the control action. It is important to notice that the considered set of contexts is itself modular. In particular, two different objects can be held simultaneously thus producing a situation which is exactly the combination of the two contexts corresponding to holding separately the two objects.

Once again the dynamical system to be controlled can be expressed as a differential equation with the structure of (1.3). However in this case the matrices M, C and g depend on the context as consequence of the fact that holding different objects results in changing the the limb dynamical parameters. Let's group all these dynamical parameters in a vector $\mathbf{p}$:

$$\mathbf{p} = \begin{bmatrix} m_i \; I_1^i \; \dots \; I_6^i \; c_x^i \; c_y^i \; c_z^i \end{bmatrix}_{i=1\dots n}^{\top}, \tag{1.5}$$

where m_i is the mass of the i^{th} link, $I_1^i, \dots, I_6^i$ represent the entries of the symmetric inertia tensor, and $\left[c_x^i, c_y^i, c_z^i\right]^{\top}$ is the center of mass position. The system to be controlled is therefore:

$$M_{\mathbf{p}}(\mathbf{q})\ddot{\mathbf{q}} + C_{\mathbf{p}}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + g_{\mathbf{p}}(\mathbf{q}) = \mathbf{u}, \tag{1.6}$$

where the notation explicitly indicates that the controlled dynamical system depends on the contexts by explicitly indicating the subscript $\mathbf{p}$.

1.3.1 Modules for handling admissible contexts

The present section answers a fundamental question which is strictly related to the problem of adapting to a continuously changing environment. In order to adapt to new contexts do we need to adapt the primitives themselves or can we just adapt the way of combining a unique and "hardwired" set of primitives? Clearly (see Section 1.2), given a context $\mathbf{p}$, a set of primitives $\{\Phi_{\mathbf{p}}^1(\mathbf{x}), \Phi_{\mathbf{p}}^2(\mathbf{x}), \dots, \Phi_{\mathbf{p}}^K(\mathbf{x})\}$ can be tailored on the specific context. An alternative solution consists in trying to find a unique set of primitives whose combination can handle all possible contexts. However, the existence of such a set of primitives cannot be guaranteed a priori and needs to be proved.

The present section shows that that such a set of primitives exists for the given set of contexts described by (1.5) (1.6). Practically, a solution is proposed to the the following problem where instead of controlling (1.3) the goal is to control (1.6) which is context dependent.

Problem 2 (Synthesis of Elementary Controls for Reaching in Different Contexts). Find a set of modules $\{\Phi^1, \dots, \Phi^K\}$ and a continuously differentiable function $\lambda(\cdot,\cdot)$, such that for every desired final state $\mathbf{q}_f$ and for every possible context $\mathbf{p}$ the input:

[5] Within this framework the controlled system is composed of the arm *and* the held object.

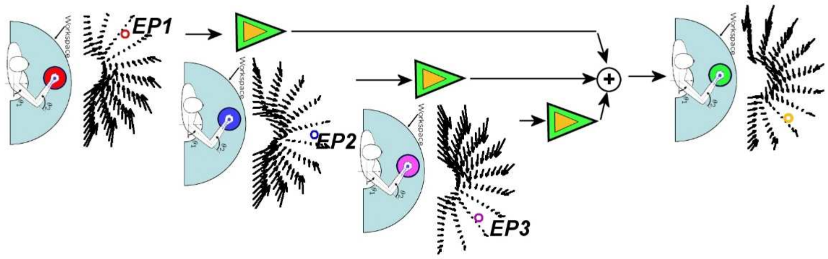

Fig. 1.2. A graphical representation of the modular reaching problem within different contexts. A finite number of reaching movements (each with a specific equilibrium point, indicated with EP, and a specific context, indicated with a different held objects) are used as primitives; these movements are represented with the associated force fields. Given an arbitrary point to be reached (represented in yellow) in a specific context (represented in green) the primitives needs to be linearly combined so as to obtain a new force field capable of reaching the desired point in the given context. Notice that the linear mixing coefficients (the triangles) are represented with green and yellow colors so as to make explicit their dependence on the desired reaching point and the current context.

$$\mathbf{u} = \sum_{k=1}^{K} \lambda_k(\mathbf{q}_f, \mathbf{p}) \Phi^k(\mathbf{q}, \dot{\mathbf{q}}) \tag{1.7}$$

steers the system (1.6) to the configuration $\mathbf{q}_f$ (see also Figure 1.2).

Obviously the proposed problem is related to the question posed in the introduction: is there a way to choose the elementary (inverse) models so as to cover all the contexts within a specified set? The answer turns out to be 'yes'. Specifically, a complete procedure for constructing a solution of problem 2 has been proposed in [Nori, 2005]. The solution turns out to have the following structure:

$$\mathbf{u} = \sum_{i=1}^{I} \sum_{j=1}^{J} \lambda_i(\mathbf{q}_f) \mu_j(\mathbf{p}) \Phi^{i,j}(\mathbf{q}, \dot{\mathbf{q}}), \tag{1.8}$$

where $\{\Phi^{1,j}, \ldots, \Phi^{I,j}\}$ is a solution to problem 1 for a specific context $\mathbf{p}^j$; minimality results can be easily extended within this framework.

1.3.2 Adaptive modules combination

Taking advantage of the results proposed in Section 1.3.1, the present section faces the problem of adaptively combining a given set of primitives to a continuously changing environment. The first part of the section, describes a method for adaptively adjusting the modules selection on the basis of performance errors. In the second part, it is shown that adaptation can be made on the basis of context related information.

1.3.3 Performance error based adaptation

In many situations, the context of the movement is not known *a priori* and therefore no previously acquired context related sensory information can be exploited. Within the proposed formulation, if the context $\mathbf{p}$ is unknown, there is no information on how to combine the primitives since the functions $\mu_j(\mathbf{p})$ cannot be evaluated. This is a consequence of the fact that the way modules are combined depends not only on the desired final position $\mathbf{q}_f$ but also on the current context $\mathbf{p}$. A possible solution consists in adaptively choosing μ_j (which are context dependent) on the basis of available data. When the only information available is the performance error[6], the estimation problem can be reformulated in terms of an adaptive control problem. It can be proved that a way to successfully reach the desired final position $\mathbf{q}_f$ consists in adaptively adjusting μ_j according to the following differential law:

$$\frac{d}{dt}\mu_j = -\mathbf{s}^\top \left[\sum_{i=1}^{I} \lambda_i(\mathbf{q}_f) \varPhi^{i,j}(\mathbf{q}, \dot{\mathbf{q}}) \right], \tag{1.9}$$

where $\mathbf{s}$ is the performance error (see [Kozlowski, 1998] for details). A mathematical proof of the system stability properties is out of the scope of the present paper and is therefore omitted. It suffices to say that, in fact, it can be proved that (1.9), together with (1.4) and (1.6), leads to a stable system.

1.3.4 Context based adaptation

The previous section have shown how performance errors can be used to adapt inverse internal models to an unknown context. This type of adaptation is useful when no a priori information is available. However, its major drawback relies in the fact that adaptation takes place only in presence of systematic errors. Moreover, every time the context is switched a certain amount of time is required in order for the adaptation process (1.9) to converge. Within certain scenarios, this adaptation process might be very long and performance errors highly undesirable. In these situations, different adaptation strategies should be adopted.

Humans exhibit an extraordinary capability of exploiting previous experiences to plan their control actions. Consider for example the problem of lifting up an object. Clearly, the movement is preplanned on the basis of some *a priori* information[7] which allows to perform actions in different contexts without systematic performance errors. How do humans create this a priori information when lifting an object that they have never lifted before?

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[6] The performance error $\mathbf{s}$ measures the difference between the desired reaching trajectory $\mathbf{q}^d$ and the actual trajectory $\mathbf{q}$. Further details can be found in [Kozlowski, 1998]

[7] A very common situation reveals that movements are preplanned on the basis of some a priori information. Consider for example the movement of lifting up an empty bottle. Sometimes humans misinterpret the context, considering the bottle full and therefore using a wrong a priori information. In these situations, the resulting movement presents an overshoot, thus revealing that a wrong a priori information has been used.

A priori information of unknown objects can be obtained on the basis of similarity. Trivially, if an unknown object is similar to a known one, then a control action suitable for the latter should be suitable also for the former. This section shows that another way to retrieve a priori information consists in interpreting an unknown object as the combination of two or more known objects. However, the way of using such an a priori information for choosing a suitable control action is nontrivial. Specifically, the modularity of the control architecture will play a fundamental role within this framework.

Modularity of forward models

As it was previously pointed out, each module is a combination of inverse and forward models. This section focuses on the forward part while leaving the discussion on the inverse part to the remainder of the chapter. In particular, it is pointed out that the modularity of contexts reflect into the modularity of the corresponding forward models. Specifically, it is shown that the forward model describing the dynamic behavior of a composed object is obtained by the linear sum of the forward models describing the component objects.

Consider the forward model associated to a limb not holding any object (1.3):

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + g(\mathbf{q}) = \mathbf{u}, \tag{1.10}$$

Let's rewrite it as follows:

$$Y(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \mathbf{u}. \tag{1.11}$$

Let's modify this model in order to take into account an externally perturbing force $\mathbf{f}_1$. We have [Murray et al., 1994]:

$$Y(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) + Y_{\mathbf{f}_1}(\mathbf{q}) = \mathbf{u}, \qquad Y_{\mathbf{f}_1}(\mathbf{q}) = J^\top(\mathbf{q})\mathbf{f}_1 \tag{1.12}$$

where J is the Jacobian matrix describing the point where the force is applied. How should (1.11) be modified when two forces $\mathbf{f}_1$ and $\mathbf{f}_2$ are simultaneously applied? Remarkably, the following holds:

$$Y(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) + Y_{\mathbf{f}_1}(\mathbf{q}) + Y_{\mathbf{f}_2}(\mathbf{q}) = \mathbf{u}, \quad Y_{\mathbf{f}_1}(\mathbf{q}) = J^\top(\mathbf{q})\mathbf{f}_1, Y_{\mathbf{f}_2}(\mathbf{q}) = J^\top(\mathbf{q})\mathbf{f}_2 \tag{1.13}$$

Interestingly, the addition of new forces reflects to the additivity if the corresponding forward models. Is a similar property preserved when considering the modularity of objects? To answer this question, let's rewrite (1.6) as follows:

$$Y_{\mathbf{p}}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \mathbf{u}. \tag{1.14}$$

Suppose that a context $\mathbf{p}_1$ corresponds to holding an object $\mathcal{O}_1$. It can be shown that [Kozlowski, 1998] the following equation holds:

$$Y_{\mathbf{p}_1}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = Y(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) + Y_{\mathcal{O}_1}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}). \tag{1.15}$$

where Y describes the forward model of the limb not holding any object and $Y_{\mathcal{O}_1}$ the forces (inertial, Coriolis and gravitational) due to the held object. The importance of this decomposition is evident when our representation of the forward dynamics is itself modular. Practically, the forward dynamics of the limb holding an object

($Y_{\mathbf{p}_1}$) can be obtained by combining the module describing the limb dynamic (Y) and the module describing the object (O_1), i.e. $Y_{\mathbf{p}_1} = Y + O_1$.

Using this property it can be shown that the effect of two different objects $\mathcal{O}_1$ and $\mathcal{O}_2$ is given by:

$$Y_{\mathbf{p}}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = Y(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) + Y_{\mathcal{O}_1}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) + Y_{\mathcal{O}_2}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}), \tag{1.16}$$

which can be interpreted as the fact that additivity of forces (1.13) is preserved when considering modular objects. The best way to exploit this compositional property consists in using a representation of the internal models which shares a similar compositionality. Remarkably the modular representation have this property. Suppose that the internal model is represented by the linear combination of the elementary modules $\Phi^1(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$, $\Phi^2(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$, $\ldots$, $\Phi^J(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}})$. Two different objects $\mathcal{O}_1$ and $\mathcal{O}_2$ (corresponding to contexts $\mathbf{p}_1$ and $\mathbf{p}_2$) can therefore be represented as follows:

$$Y_{\mathcal{O}_1}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \sum_{j=1}^{J} \Psi_j(\mathbf{p}_1)\Phi^j(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}), \tag{1.17}$$

$$Y_{\mathcal{O}_2}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \sum_{j=1}^{J} \Psi_j(\mathbf{p}_2)\Phi^j(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}). \tag{1.18}$$

Given (1.16), the composition of the two objects is represented by a new internal model which is described by the same elementary modules. Specifically we have:

$$Y_{\mathcal{O}}(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}) = \sum_{j=1}^{J} \left[\Psi_j(\mathbf{p}_1) + \Psi_j(\mathbf{p}_2)\right] \Phi^j(\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}), \tag{1.19}$$

where the new object $\mathcal{O}$ and the associated context $\mathbf{p}$ interpreted as the composition of $\mathbf{p}_1$ and $\mathbf{p}_2$ is internally represented by the composition of the associated internal model. Moreover, the modules mixing coefficients satisfy the following:

$$\Psi_j(\mathbf{p}) = \Psi_j(\mathbf{p}_1) + \Psi_j(\mathbf{p}_2) \qquad j = 1 \ldots J, \tag{1.20}$$

which implies that also this coefficients share the compositional property.

Modularity of inverse models

In the previous section it was shown that the additivity of forces implies the additivity of forward models. Exploiting this property, the advantages of modular representations of forward internal models were discussed in details. Interestingly, a similar result holds for inverse internal models. Specifically, following the procedure proposed in [Nori and Frezza, 2005], it can be shown that the modular inverse model (1.8) posses the same property described by (1.20). In particular, if a context $\mathbf{p}$ can be interpreted as the combination of the contexts $\mathbf{p}_1$ and $\mathbf{p}_2$, then the following holds:

$$\mu_j(\mathbf{p}) = \mu_j(\mathbf{p}_1) + \mu_j(\mathbf{p}_2) \qquad j = 1 \ldots J. \tag{1.21}$$

Proving the correctness of the above equation falls outside the scope of this chapter and will be therefore omitted. The complete proof starts from (1.20) and uses the fact that the inverse model (1.8) depends linearly on the forward model (see [Nori and Frezza, 2005, Nori, 2005]).

1.4 Experimental results

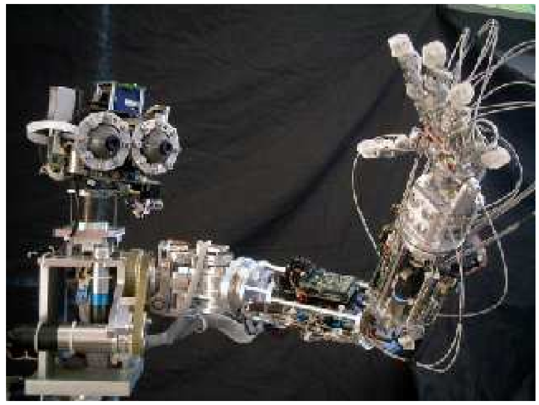

Fig. 1.3. James, a humanoid upper torso.

In this section we describe some experimental results that have been obtained by implementing a very simple modular control strategy on our humanoid robot (see Figure 1.3). The purpose of this section is to show that equation (1.21) holds and can be used to predict the inverse model suitable in a context which is the combination of previously encountered contexts.

The experimental scenario is very simple. The robot picks up a first object and explores its dynamical properties by learning a suitable inverse model. After this phase a second object is picked up and similarly explored. Finally, a third object, nominally the combination of the previous two, is picked up and two alternatives are compared: the first consists in exploring this third object as if it were a new one; the second instead exploits the fact that this new object is the combination of the two previously learned objects.

In order to simplify the scenario, we consider a very simple inverse model corresponding to the motor torques necessary to compensate for the gravitational forces. Obviously these torques depend on the dynamical properties of the grasped object. The objects used in the experiment are two cylinders of different weights. The objects can be inserted one into the other so as to form a combined object.

During the exploration phase the robot randomly moves around its arm while holding the objects. During this phase the robots uses the first four degrees of freedom of the arm (three in the shoulder and one in the elbow). At each rest position, the motor torques necessary to support the arm are collected. The inverse model maps a given arm position $\mathbf{q}$ into the torques $\mathbf{u}$ necessary to counterbalance the gravitational forces. This model has been represented modularly as follows[8]:

[8] In this specific example we are mainly interested in the context-dependent part of the inverse model. This is the reason why (1.22) differs from (1.8) where the goal dependent part was also included. Moreover, since gravity is compensated at rest we also have $\dot{\mathbf{q}} = 0$.

$$\mathbf{u} = \sum_{j=1}^{J} \mu_j(\mathbf{p})\Phi^j(\mathbf{q}). \tag{1.22}$$

The basis functions $\Phi^j(\mathbf{q})$ can be chosen freely. In the results presented here we followed a parametric strategy as proposed in [Kozlowski, 1998]. Alternative strategies would have been a support vector machine and radial basis function representations. Given a collection of training pairs $(\mathbf{u}_k, \mathbf{q}_k)_{k=1}^{K}$ describing the torques to compensate gravity in a given context $\mathbf{p}$, the values for $\mu_j(\mathbf{p})$ are estimated by solving a minimum least squares problem:

$$[\hat{\mu}_1(\mathbf{p}) \ldots \hat{\mu}_J(\mathbf{p})] = \min_{\mu} \sum_{k=1}^{K} \left\| \mathbf{u}_k - \sum_{j=1}^{J} \mu_j \Phi^j(\mathbf{q}_k) \right\|^2 . \tag{1.23}$$

In order to validate the estimation process a set of testing pairs $(\mathbf{u}_l, \mathbf{q}_l)_{l=1}^{L}$ are used to compute the prediction errors $\mathbf{e}_l$:

$$\mathbf{e}_l = \mathbf{u}_l - \sum_{j=1}^{J} \hat{\mu}_j(\mathbf{p})\Phi^j(\mathbf{q}_l). \tag{1.24}$$

The resulting mean square errors (MSE) have been computed and reported in Table 1.1. In this table three contexts are considered: first object held ($\mathbf{p}_1$), second object ($\mathbf{p}_2$) and both objects ($\mathbf{p}_{1\&2}$). Finally the last line of the table corresponds to the mean square error obtined if instead of using the training data to estimate $\mu_j(\mathbf{p}_{1\&2})$ we simply use the property (1.21) i.e. $\mu_j(\mathbf{p}_{1\&2}) = \mu_j(\mathbf{p}_1) + \mu_j(\mathbf{p}_2)$. Even thought the performances in this last case are not optimal, the obtained results are still interesting since they have been obtained without any learning on the training data.

Table 1.1. Mean square error results. The training set is composed of 300 samples, i.e. $K = 300$. The test set of 100 samples, i.e. $L = 100$.

Context	Training set MSE	Testing set MSE
$\mathbf{p}_1$	0.5392	0.6086
$\mathbf{p}_2$	0.5082	0.5638
$\mathbf{p}_{1\&2}$	0.6143	0.5770
$\mathbf{p}_1 + \mathbf{p}_2$	-	0.7458

1.5 Open issues

Modern artificial systems fail to adapt to the variety of situations which can be encountered in everyday life. Nowadays, even the most sophisticated machines are designed to work and operate within highly structured environment. Therefore, building highly interactive and adaptive systems capable of operating in a “human-like” environment seems to be one of fundamental challenge in the field of robotics.

Within this context, recent studies have proposed the importance of modular control structures. Specifically, it has been argued that the big advantage of modularity relies in the possibility of combining previous experience to handle new situations. As to this concern, in this chapter it was shown that the advantages of modularity can be exploited by interpreting the current context as the combination of previously experienced contexts. Moreover, it was pointed out that the variety of contexts in everyday life is such that every brute force adaptive strategy have to deal with the curse of dimensionality. As a consequence, innovative adaptive strategies which scale linearly with the number of contexts need to be considered. We believe that modularity possess this property but we also believe that many open issues remain before building an artificial system capable of exploiting its potentialities.

Specifically, in the present chapter we focused on motor control modularity stressing on its capability of adapting to modular contexts. However, we did not mention any result on how to retrieve context related sensory information from the environment. Clearly, this information requires the use of sensors capable of extracting salient features from the surrounding environment. Therefore, the implementation of these ideas seems to fit perfectly within the field of humanoid robotics. We are currently using a humanoid upper torso, James (see figure 1.3), as a testbed to explore the potentialities of these ideas. Currently, the platform is equipped with three kind of exteroceptive sensors: vision, force (Figure 1.4) and touch sensors (Figure 1.5). Potentially these sensors can be used to retrieve context related information from the environment. Remarkably, representing this information modularly still remains an open issue to be further investigated.

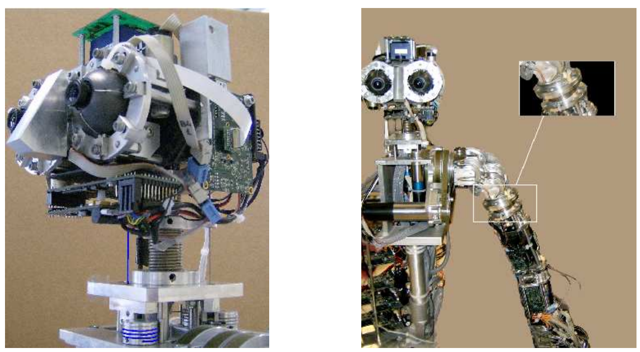

Fig. 1.4. Pictures of the vision (left) and force (right) sensors mounted on James. Vision is achieved with a stereo pair of digital cameras. External forces are instead measured with a six degrees of freedom force sensor placed in the forearm of the robot.

1.6 Future works

Currently we are exploring the potentialities of the approach within the context of manipulation. Manipulating objects is a complicated task because of the high

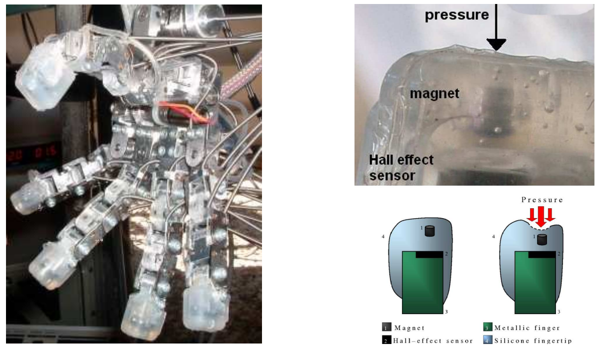

Fig. 1.5. Picture (left) and schematics (right) of the touch sensors mounted on the hand of James. Each fingertip has a couple of sensors based on a Hall effect technology. Eight more touch sensors are mounted on the phalanxes.

number of hand degrees of freedom and because of the multiplicity of objects which can be manipulated. Clearly, an exploration approach which scales exponentially with the complexity will fail within this context. We are therefore exploring the modular approach as a technique to simplify the complexity of the task. Remarkably there is evidence supporting the idea that a handful of motor synergies can be used in manipulation tasks. Our current research is not only investigating the potentialities of a smart mechanical design [Brown and Asada, 2007] but also the role of synergies in human manipulation [Santello et al., 1998, Baud-Bovy et al., 2005].

1.7 Conclusions

Modular control structures are appealing since there exist contexts which can be modular as well. In order to investigate this concept, the present work considered a simple movement (moving the arm towards a target) within different contexts (handling different objects). Intuitively, a modular control structure is best suited to operate within modular contexts. In the specific problem of moving the arm while holding different objects, it was shown that the system dynamics are modular themselves. Taking advantage of this property it was demonstrated that a modular control structure is capable of handling multiple contexts. Finally, two ways to adaptively combine the modules has been proposed. A first adaptation is based on performance errors only. A second adaptation, which relies on the first, is used to combine the modules in unexperienced contexts which can be interpreted as the combination of previously experienced contexts.

Acknowledgments

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A Minimum Number of Motion Primitives

A.1 Lower Bound on the Number of Elementary Controls

In this Section it is proved that any given solution of Prb. 1 is composed by at least n elementary controls, i.e. $K \geq n$. Before giving the main result, it is important to recall the following lemma, claiming the injectivity of the function λ. In the following the set of reachable states is denoted $\mathcal{X} \subseteq \mathbb{R}^n$.

Lemma 1. *Let $\{\Phi^1, \ldots, \Phi^K\}$ and $\lambda : \mathcal{X} \to \mathbb{R}^K$ be a solution to Problem 1. Then $\lambda(\mathbf{x}_f)$ is injective.*

Proof. Suppose by contradiction that there exist $\mathbf{x}_f^1$ and $\mathbf{x}_f^2$ such that $\mathbf{x}_f^1 \neq \mathbf{x}_f^2$ but $\lambda(\mathbf{x}_f^1) = \lambda(\mathbf{x}_f^2)$. Define:

$$\mathbf{u}^1 \triangleq \sum_{k=1}^{K} \lambda_k(\mathbf{x}_f^1)\Phi^k(\mathbf{x}), \tag{1.25}$$

$$\mathbf{u}^2 \triangleq \sum_{k=1}^{K} \lambda_k(\mathbf{x}_f^2)\Phi^k(\mathbf{x}). \tag{1.26}$$

Under the given assumption $\mathbf{u}^1 = \mathbf{u}^2$, but this contradicts the fact that $\mathbf{u}^1$ drives the system to $\mathbf{x}_f^1 \neq \mathbf{x}_f^2$. By contradiction, this proves that λ is injective. $\diamondsuit$

Proposition 1 *Let $\{\Phi^1, \ldots, \Phi^K\}$ and $\lambda : \mathcal{X} \to \mathbb{R}^K$ be a solution to Problem 1. Then $K \geq n$.*

Proof. Using Lemma 1, we have that $\lambda(\cdot)$ is a continuously differentiable injective function from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^K$. It can be proved that this implies $K \geq n$ (see [Boothby, 2002] for details). $\diamondsuit$

A.2 Minimum number of elementary controls

It was shown that we need at least n primitives to preserve controllability. It is here proved that n are not sufficient under suitable assumptions. Let's consider an affine nonlinear system of type (1.1). Let $\{\Phi^1, \ldots, \Phi^K\}$, $\lambda : \mathcal{X} \to \mathbb{R}^K$ be a solution to Problem 1. Moreover, define $\mathcal{X}_{\text{eq}} \subset \mathcal{X}$ to be the set of equilibrium points with $\mathbf{u} = 0$, i.e. $\mathbf{x}_f \in \mathcal{X}_{\text{eq}}$ if and only if $f(\mathbf{x}_f) = 0$. The proof of the main result requires the following additional assumption.

(HP). $\mathcal{X}_{\text{eq}} \neq \emptyset$ and $\forall \mathbf{x}_f \in \mathcal{X}_{\text{eq}}$ the solution of the following differential equation:

$$\begin{cases} \dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x}) \sum_{k=1}^{K} \lambda_k(\mathbf{x}_f)\Phi^k(\mathbf{x}, t) \\ \mathbf{x}(0) = \mathbf{x}_f \end{cases} . \tag{1.27}$$

is $\mathbf{x}(t) \equiv \mathbf{x}_f$, $\forall t \in [0, T]$. Basically, this is equivalent to requiring that if we start at an equilibrium point and we want to reach the same equilibrium point, then this is accomplished leaving the system in the same position for the entire time interval.

Theorem 2. *Let* $\{\Phi^1, \dots, \Phi^K\}$, $\lambda : \mathcal{X} \to \mathbb{R}^K$ *be a solution to Problem 1 with dynamics given by (1.1). Moreover, assume that for the given solution, (HP) holds. Then* $K \geq n+1$.

Proof. Suppose by contradiction that $K < n+1$. Since it was shown that $K \geq n$, only one possibility is left out: $K = n$. Define:

$$\Phi(\mathbf{x}, t) \triangleq \left[\Phi^1(\mathbf{x}, t) \ \dots \ \Phi^n(\mathbf{x}, t)\right] \qquad \lambda(\mathbf{x}_f) \triangleq \left[\lambda_1(\mathbf{x}_f) \ \dots \ \lambda_n(\mathbf{x}_f)\right]^\top, \quad (1.28)$$

so that the following holds:

$$\sum_{k=1}^{n} \lambda_k(\mathbf{x}_f)\Phi^k(\mathbf{x}, t) = \Phi(\mathbf{x}, t)\lambda(\mathbf{x}_f). \quad (1.29)$$

Since the proposed primitives solve Prb. 1, the following system:

$$\begin{cases} \dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\Phi(\mathbf{x}, t)\lambda(\mathbf{x}_f) \\ \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X} \end{cases}, \quad (1.30)$$

converges to $\mathbf{x}_f$, $\forall \mathbf{x}_f \in \mathcal{X}$ regardless of the initial condition. Consider $\mathbf{x}_f \in \mathcal{X}_{\text{eq}}$ and $\mathbf{x}(0) = \mathbf{x}_f$. Using (HP) in (1.30), we get:

$$g(\mathbf{x}_f)\Phi(\mathbf{x}_f, t)\lambda(\mathbf{x}_f) = 0, \qquad \forall t \in [0, T]. \quad (1.31)$$

Now, let's use the fact that the image of $\mathcal{X}$ under $\lambda : \mathcal{X} \to \mathbb{R}^n$ is an open set in $\mathbb{R}^n$. This is consequence of the fact that λ is injective and $\mathcal{C}^1(\mathcal{X})$ with $\mathcal{X}$ open (see [Boothby, 2002] for details). If $\text{Im}_\lambda(\mathcal{X})$ is open and $\lambda(\mathbf{x}_f) \in \text{Im}_\lambda(\mathcal{X})$, we can always find $\mu \neq 1$ such that $\mu\lambda(\mathbf{x}_f) \in \text{Im}_\lambda(\mathcal{X})$; we only need to take $|1-\mu|$ small enough. Therefore, there always exists $\tilde{\mathbf{x}}_f \in \mathcal{X}$, such that $\lambda(\tilde{\mathbf{x}}_f) = \mu\lambda(\mathbf{x}_f)$. According to the given hypothesis, the following system:

$$\begin{cases} \dot{\mathbf{x}} = f(\mathbf{x}) + g(\mathbf{x})\Phi(\mathbf{x}, t)\lambda(\tilde{\mathbf{x}}_f) \\ \mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{X} \end{cases}, \quad (1.32)$$

should converge to $\tilde{\mathbf{x}}_f$ regardless of the initial condition. However, take $\mathbf{x}_0 = \mathbf{x}_f$, and observe that the corresponding trajectory is given by $\mathbf{x}(t) \equiv \mathbf{x}_f$, $\forall t \in [0, T]$ since we have:

$$\dot{\mathbf{x}} = f(\mathbf{x}_f) + g(\mathbf{x}_f)\Phi(\mathbf{x}_f, t)\lambda(\tilde{\mathbf{x}}_f) = \mu g(\mathbf{x}_f)\Phi(\mathbf{x}_f, t)\lambda(\mathbf{x}_f) \overset{(1.31)}{=} 0 \quad \forall t \in [0, T].$$

Therefore, we have found an initial condition for (1.32) that is not driven to $\tilde{\mathbf{x}}_f$. This is in contradiction with our hypothesis. ♢

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